JOURNAL OF ALGEBRA **193**, 122–143 (1997)
ARTICLE NO. JA966963

The Structure of Some Permutation Modules for the Symmetric Group of Infinite Degree

Darren G. D. Gray

School of Mathematics, University of East Anglia, Norwich NR4 7TJ, England

Received June 20, 1996

Suppose that Ω is an infinite set and k is a natural number. Let $[\Omega]^k$ denote the set of all k-subsets of Ω and let F be a field. In this paper we study the $FSym(\Omega)$-submodule structure of the permutation module $F[\Omega]^k$. Using the representation theory of finite symmetric groups, we show that every submodule of $F[\Omega]^k$ can be written as an intersection of kernels of certain $FSym(\Omega)$-homomorphisms $F[\Omega]^k \to F[\Omega]^l$ for $0 \leq l < k$, and give a simple algorithm to determine the complete submodule structure of $F[\Omega]^k$. © 1997 Academic Press

1. INTRODUCTION AND NOTATION

1.1. *Introduction*

In this paper we shall investigate the submodule structure of certain permutation modules for the symmetric group $Sym(\Omega)$, where Ω is an infinite set. If k is a natural number, then we can form $F[\Omega]^k$, the vector space over a field F with basis elements the subsets of size k from Ω. This vector space has a natural $Sym(\Omega)$-action, giving $F[\Omega]^k$ the structure of an $FSym(\Omega)$-module. When the field F has characteristic zero, the submodule structure of $F[\Omega]^k$ is explicitly known (see [1]). The main result of this paper is an algorithm which enables us to effectively compute the submodule structure of $F[\Omega]^k$ when F is of prime characteristic p. The algorithm presented here basically consists of checking whether or not p divides certain binomial coefficients. However, many of the results presented in this paper are independent of the characteristic of the field, so we can compute the submodule structure of $F[\Omega]^k$ for any field F.

As is well known, $F[\Omega]^k$ is $FSym(\Omega)$-isomorphic to M^λ, a module defined using a particular partition λ of Ω. For the case when Ω is a finite

0021-8693/97 $25.00

set, a great deal is known about M^λ and its submodules for arbitrary partitions λ, and we refer the reader to the works of James (see, for example, [4–6]). We will make use of these finite case results to prove analogous results when Ω is infinite.

In Section 1.2 we introduce our notation and definitions, in particular, we introduce our concept of an infinite partition. Most of the definitions here have been adapted from their finite counterparts, and the formal definitions of these can be found in [6]. The most important definition is that of the Specht module S^λ, a certain submodule of M^λ.

Section 2 contains some general results about M^λ and S^λ when λ is an infinite partition. These are analogues or consequences of results for finite Ω.

Section 3 introduces the module $F[\Omega]^k$ and the connection with partitions. Most of the results needed to compute the submodule structure of $F[\Omega]^k$ are found in this section. We show here that the Specht module for $F[\Omega]^k$ is irreducible whatever the characteristic of the field, and we also show that the composition factors of $F[\Omega]^k$ are precisely the Specht modules of $F[\Omega]^l$ for $l = 0, 1, \ldots, k$, each appearing with multiplicity one. We describe each submodule of $F[\Omega]^k$ as an intersection of kernels of certain $FSym(\Omega)$-homomorphisms.

We look at some special cases in Section 4, and then proceed in Section 5 to give a description of an algorithm which computes the submodule structure of $F[\Omega]^k$ when the characteristic of the field is a prime p.

This paper does not seem to overlap with the existing body of work studying representations of the finitary infinite symmetric group $S(\infty)$, that is, the group of all finite permutations of a countably infinite set. Extensive work has been undertaken to classify irreducible representations of $S(\infty)$ over the complex numbers. This work was initiated by Thoma [9] who described the finite characters of $S(\infty)$. Since then, other classes of representations have been discovered; see, for example, the works of Lieberman [7], Ol'shanskii [8], Vershik and Kerov [10–12], and Hirai [3]. This paper takes a different approach, starting with a particular permutation representation (defined over an arbitrary field) and finding all the irreducible representations associated with this permutation representation.

1.2. *Definitions and Notation*

We begin by introducing our notion of a *partition*.

DEFINITION 1.1. Let Λ be any set and r a natural number. Then we say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ is a *partition* of Λ, written $\lambda \vdash \Lambda$, if there exist pairwise disjoint subsets $A_1, A_2, \dots, A_r$ of Λ such that:

1. $\lambda_i = |A_i|$ $(i = 1, 2, \dots, r)$;
2. λ_i is finite for $i = 2, \dots, r$; and
3. $\bigcup_{i=1}^{r} A_i = \Lambda$.

If further, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > 0$, then we say that the partition is *proper*, otherwise it is *improper*. It will be assumed throughout that all partitions are proper unless stated otherwise. We say that the partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ has *r parts*. If $|\Lambda| = n < \infty$ then we say that λ is a *partition of n*, written $\lambda \vdash n$, and we have that $\sum_{i=1}^{r} \lambda_i = n$.

Without loss of generality, if Λ is infinite, we will take Λ to be $\mathbf{N}$, the natural numbers, and if Λ is finite of size n, we will take Λ to be the set $\{1, 2, \dots, n\}$.

We shall occasionally refer to a partition of n as a *finite partition* and a partition of $\mathbf{N}$ as an *infinite partition*.

We can represent partitions diagrammatically. For example, if we consider the partition $\lambda = (4, 2, 2, 1)$ of 9, then the diagram of λ is

$$[\lambda] = \begin{array}{cccc} * & * & * & * \\ * & * & & \\ * & * & & \\ * & & & \end{array}$$

We can also represent infinite partitions in this way. For example, the diagram of the partition $\lambda = (\infty - 8, 5, 2, 1)$ is

$$[\lambda] = \begin{array}{cccccccc} * & * & * & * & * & * & * & \cdots \\ * & * & * & * & * & & & \\ * & * & & & & & & \\ * & & & & & & & \end{array}$$

Let λ be any partition. Then by replacing each entry of the diagram of λ by an element of Λ, allowing no repeats and using all the elements of Λ, we obtain a *λ-tableau*.

EXAMPLE 1.2. The following is a $(\infty - 4, 3, 1)$-tableau:

$$\begin{array}{ccccccc} 4 & 8 & 5 & 3 & 6 & 9 & \cdots \\ 7 & 1 & 2 & & & & \\ 10 & & & & & & \end{array}$$

We have that $Sym(\Lambda)$ acts transitively on the set of λ-tableaux in the natural way; if t is a λ-tableau, $\pi \in Sym(\Lambda)$, and α the ijth node of t, then $\alpha\pi$ is the ijth node of $t\pi$.

For a λ-tableau t, we define its *row stabilizer*, R_t, to be that subgroup of $Sym(\Lambda)$ which fixes the rows of t setwise, and similarly its *column stabilizer*, C_t, to be that subgroup of $Sym(\Lambda)$ which fixes the columns of t setwise. Note that, since λ_2 is finite, C_t is a finite subgroup of $Sym(\Lambda)$.

We can define an equivalence relation on the set of λ-tableaux by $t_1 \sim t_2$ if and only if $t_1\pi = t_2$ for some $\pi \in R_{t_1}$, and we define a *tabloid* $\{t\}$ to be the equivalence class of t with respect to this equivalence relation. So a tabloid can be considered as a tableau with unordered row entries. $Sym(\Lambda)$ acts transitively on the λ-tabloids by $\{t\}\pi = \{t\pi\}$.

If we now let F be an arbitrary field, and let M^λ be the vector space over F whose basis elements are the various λ-tabloids, then the action of $Sym(\Lambda)$ on the λ-tabloids turns M^λ into an $FSym(\Lambda)$-module.

Associated with each λ-tableau t we have a *polytabloid*, e_t, defined by

$$e_t = \{t\} \sum_{\pi \in C_t} (\operatorname{sgn} \pi)\pi,$$

where

$$\operatorname{sgn} \pi = \begin{cases} +1 & \text{if } \pi \text{ is an even permutation,} \\ -1 & \text{if } \pi \text{ is an odd permutation.} \end{cases}$$

The $FSym(\Lambda)$-submodule of M^λ spanned by the various λ-polytabloids is of great importance, and is called the *Specht module* for the partition λ, and is denoted by S^λ. The following useful result follows easily from the transitivity of $Sym(\Lambda)$ on the tabloids.

PROPOSITION 1.3. *S^λ is a cyclic $FSym(\Lambda)$-module, generated by any one polytabloid.*

Now there is a natural bilinear form on M^λ defined by

$$\langle \{t_1\}, \{t_2\} \rangle = \begin{cases} 1 & \text{if } \{t_1\} = \{t_2\}, \\ 0 & \text{if } \{t_1\} \neq \{t_2\} \end{cases}$$

for all λ-tabloids $\{t_1\}, \{t_2\}$, and, of course, extended linearly to the whole of M^λ.

This bilinear form is clearly symmetric, $Sym(\Lambda)$-invariant, and nonsingular. We let $\perp$ denote orthogonality with respect to this form.

If V is an $FSym(\Lambda)$-submodule of M^λ, then we write $V \leq M^\lambda$, and say that V is a *submodule* of M^λ if the context is clear. If V is a proper $FSym(\Lambda)$-submodule of M^λ, i.e., $V \neq M^\lambda$, then we write $V < M^\lambda$.

DEFINITION 1.4. If $v \in M^\lambda$, then v is a unique linear combination of tabloids; we say that the tabloid $\{t\}$ is *involved* in v if the coefficient of $\{t\}$

in v is nonzero. We define the *support* of v, $Supp(v)$, to be the set of tabloids involved in v, and the *weight* of v, denoted $weight(v)$, to be the cardinality of $Supp(v)$. Note that the weight of any element of M^λ is finite.

2. GENERAL RESULTS

Throughout this section let Ω be any infinite set (and again, for simplification purposes, we can, without loss of generality, identify Ω with the set of natural numbers). Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of Ω. We will work over an arbitrary field F.

Undoubtedly, the most important result is the following theorem which involves both the bilinear form defined earlier and the Specht module:

THEOREM 2.1 (The Submodule Theorem). *If U is a submodule of M^λ, then either $U \geq S^\lambda$ or $U \leq (S^\lambda)^\perp$.*

This powerful result is due to James, who proved it for finite partitions (see 4.8 in [6]). However, the proof can easily be adapted to deal with infinite partitions. We need the following terminology to allow us to restrict to finite partitions and use the results already available.

DEFINITION 2.2. Let $\{t\} \in M^\lambda$. Then for $i = 1, 2, \dots, r$, let $R_i(\{t\})$ be the ith row of $\{t\}$. Now let $n \in \mathbf{N}$ be such that $n > 2\lambda_2 + \lambda_3 + \cdots + \lambda_r$. Then we define $M^\lambda[n]$ to be the following subspace of M^λ:

$$M^\lambda[n] := \left\langle \{t\} \in M^\lambda\colon \bigcup_{i=2}^{r} R_i(\{t\}) \subseteq \{1, \dots, n\} \right\rangle_F.$$

Let $G[n]$ be the setwise stabilizer of $\{1, 2, \dots, n\}$ in $Sym(\Omega)$, so in fact $G[n] = Sym(\{1, 2, \dots, n\}) \times Sym(\Omega \setminus \{1, 2, \dots, n\})$. Let λ' be the partition given by $\lambda' = (n - [\lambda_2 + \cdots + \lambda_r], \lambda_2, \dots, \lambda_r)$. Then $G[n]$ acts on $M^{\lambda'}$: any $g \in G[n]$ can be written as $g = hk$, where $h \in Sym(n)$ and $k \in Sym(\Omega \setminus \{1, 2, \dots, n\})$. So if $x \in M^{\lambda'}$, then the action of $G[n]$ on x is given by $xg = xh$. Thus $G[n]$ acts on $M^{\lambda'}$ in the same way that $Sym(n)$ does, and clearly the $FG[n]$-submodules of $M^{\lambda'}$ and the $FSym(n)$-submodules of $M^{\lambda'}$ coincide.

Of course, this generalises to arbitrary submodules of M^λ: if $U \leq M^\lambda$, then $U[n]$ is that $FG[n]$-submodule of U defined by $U[n] := U \cap M^\lambda[n]$. When U is the Specht module, S^λ, of M^λ, then $S^\lambda[n]$ is isomorphic to $S^{\lambda'}$, where λ' is the finite partition above. It is clear that the $FG[n]$-submodule lattice of $M^\lambda[n]$ and the $FSym(n)$-submodule lattice of $M^{\lambda'}$ are the same. Thus we will frequently make no distinction between $M^\lambda[n]$ and $M^{\lambda'}$ (or between submodules of $M^\lambda[n]$ and $M^{\lambda'}$).

We will characterise the Specht module S^λ of M^λ as the intersection of kernels of certain $FSym(\Omega)$-homomorphisms. The following definition is based on concepts introduced in [4].

DEFINITION 2.3. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of Ω. Let i be some nonnegative integer satisfying $0 \le i < r$, let $\delta \in \mathbf{N}$ be such that $0 \le \delta \le \lambda_{i+1}$, and let the partition $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ of Ω be given by: $\mu_j = \lambda_j$ for $j \ne i,\ i+1$; $\mu_{i+1} = \lambda_{i+1} - \delta$; and $\mu_i = \lambda_i + \delta$. Note that μ may be improper. Let $\{t\}$ be any λ-tabloid. Then define $\hat{\psi}_{i,\delta}\colon M^\lambda \to M^\mu$ by

$$\hat{\psi}_{i,\delta}\colon \{t\} \mapsto \begin{cases} \text{the sum of } \mu\text{-tabloids } \{t'\}, \text{ where } \{t'\} \text{ agrees with } \{t\} \text{ on all} \\ \text{except the } i\text{th and } (i+1)\text{th rows, the } i\text{th row of } \{t'\} \\ \text{consisting of the } i\text{th row of } \{t\} \text{ together with } \delta \text{ elements} \\ \text{from the } (i+1)\text{th row of } \{t\}. \end{cases}$$

It is clear that $\hat{\psi}_{i,\delta}$ is an $FSym(\Omega)$-module homomorphism.

Thus $\hat{\psi}_{i,\delta}$ can be viewed as a map which moves δ elements up from the $(i+1)$th row of a tabloid on which it is acting to the ith row.

For $n \in \mathbf{N}$, we denote the restriction of $\hat{\psi}_{i,\delta}$ to $M^\lambda[n]$ by $\hat{\psi}^n_{i,\delta}$. So from Theorem 9.3 in [4] we have

$$S^\lambda[n] = \bigcap_{i=1}^{r-1} \bigcap_{\delta=1}^{\lambda_{i+1}} \ker \hat{\psi}^n_{i,\delta}.$$

We generalise this to S^λ:

THEOREM 2.4. *Let Ω be any infinite set and let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of Ω. Then*

$$S^\lambda = \bigcap_{i=1}^{r-1} \bigcap_{\delta=1}^{\lambda_{i+1}} \ker \hat{\psi}_{i,\delta}.$$

Proof. Let $x \in \bigcap_i \bigcap_\delta \ker \hat{\psi}_{i,\delta}$. Then, for large enough n, we have that $x \in \bigcap_i \bigcap_\delta \ker \hat{\psi}^n_{i,\delta}$. But $\bigcap_i \bigcap_\delta \ker \hat{\psi}^n_{i,\delta} = S^\lambda[n]$, and since $S^\lambda[n]$ is naturally contained in S^λ, we have that $x \in S^\lambda$. Therefore $\bigcap_i \bigcap_\delta \ker \hat{\psi}_{i,\delta} \le S^\lambda$.

Now let t be any λ-tableau, and let e_t be the corresponding polytabloid in S^λ. Then for large enough n, we have that $e_t \in S^\lambda[n]$. Thus, by the finite version of this theorem, $e_t \in \bigcap_{i=1}^{r-1} \bigcap_{\delta=1}^{\lambda_{i+1}} \ker \hat{\psi}^n_{i,\delta}$. This completes the proof. ∎

We have already noted that the map $\hat{\psi}_{i,\delta}\colon M^\lambda \to M^\mu$ can be viewed as a map which moves δ elements up from the $(i+1)$th row of a tabloid on which it is acting to the ith row. We now define a map from M^μ to M^λ which moves elements down a row.

DEFINITION 2.5. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of Ω. Let i be such that $1 < i < r$ and let $\delta \in \mathbf{N}$ satisfy $0 \le \delta \le \lambda_{i+1}$. Let the partition $\mu = (\mu_1, \mu_2, \dots, \mu_r)$ of Ω be given by: $\mu_j = \lambda_j$ for $j \ne i, i+1$; $\mu_{i+1} = \lambda_{i+1} - \delta$; and $\mu_i = \lambda_i + \delta$ (note that μ may be improper). Then define $\phi_{i,\delta}\colon M^\mu \to M^\lambda$ by

$$\phi_{i,\delta}\colon \{t\} \mapsto \begin{cases} \text{the sum of } \lambda\text{-tabloids } \{t'\}, \text{ where } \{t'\} \text{ agrees with } \{t\} \text{ on all} \\ \text{except the } i\text{th and } (i+1)\text{th rows, the } (i+1)\text{th row of } \{t'\} \\ \text{consisting of the } (i+1)\text{th row of } \{t\} \text{ together with } \delta \\ \text{elements from the } i\text{th row of } \{t\} \end{cases}$$

for any μ-tabloid $\{t\}$. Yet again it is clear that $\phi_{i,\delta}$ is an $FSym(\Omega)$-module homomorphism.

Thus $\phi_{i,\delta}$ can be viewed as a map which moves δ elements down from the ith row of a tabloid on which it is acting to the $(i+1)$th row.

Remark. Note that $\phi_{i,\delta}$ is undefined for $i = 1$, because there are infinitely many ways of choosing δ elements from the first row of any μ-tabloid. However, if as usual we denote the restriction of $\phi_{i,\delta}$ to $M^\mu[n]$ by $\phi^n_{i,\delta}$ where we insist that im $\phi^n_{i,\delta} \le M^\lambda[n]$, then $\phi^n_{i,\delta}$ is in fact defined for $i = 1$ also.

The next result is again due to James, and deals with the characterisation of the module orthogonal to the Specht module in the finite case (see Corollary 3 in [5]).

THEOREM 2.6. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of Ω, let $n \in \mathbf{N}$, and consider the ϕ-maps as defined above. Then*

$$(S^\lambda[n])^{\perp_n} = \sum_{i=1}^{r-1} \sum_{\delta=1}^{\lambda_{i+1}} \operatorname{im} \phi^n_{i,\delta},$$

where $\perp_n$ denotes orthogonality in $M^\lambda[n]$.

We would like a similar characterisation for $(S^\lambda)^\perp$, and indeed we have the following:

THEOREM 2.7. *Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ be a partition of Ω with $r > 2$ and consider the ϕ-maps as defined above. Then*

$$(S^\lambda)^\perp = \sum_{i=2}^{r-1} \sum_{\delta=1}^{\lambda_{i+1}} \operatorname{im} \phi_{i,\delta}.$$

Proof. Note firstly that $S^\lambda = \bigcup_{n \geq n_0} S^\lambda[n]$, for some n_0 large enough (for example, $n_0 \geq 2\lambda_2 + \lambda_2 + \cdots + \lambda_r$). Then it is clear that $(S^\lambda)^\perp = \bigcap_{n \geq n_0} (S^\lambda[n])^\perp$.

Now $(S^\lambda[n])^\perp = (S^\lambda[n])^{\perp_n} \oplus M^\lambda[\Omega \setminus \{1, 2, \ldots, n\}]$ and so

$$(S^\lambda)^\perp = \bigcap_{n \geq n_0} (S^\lambda[n])^{\perp_n} \oplus M^\lambda[\Omega \setminus \{1, 2, \ldots, n\}]$$

$$= \bigcap_{n \geq n_0} \left[\left(\sum_{i=1}^{r-1} \sum_{\delta=1}^{\lambda_{i+1}} \operatorname{im} \phi^n_{i,\delta} \right) \oplus M^\lambda[\Omega \setminus \{1, 2, \ldots, n\}] \right].$$

Now, to simplify notation, let

$$\Sigma^n(1) := \sum_{\delta=1}^{\lambda_{i+1}} \operatorname{im} \phi^n_{1,\delta},$$

$$\Sigma^n(2) := \sum_{i=2}^{r-1} \sum_{\delta=1}^{\lambda_{i+1}} \operatorname{im} \phi^n_{i,\delta},$$

$$\Sigma(2) := \sum_{i=2}^{r-1} \sum_{\delta=1}^{\lambda_{i+1}} \operatorname{im} \phi_{i,\delta}.$$

So what we want to prove is $(S^\lambda)^\perp = \Sigma(2)$.

Let $x \in \Sigma(2)$. Then, for large enough n we have $x \in \Sigma^n(2)$. But now $\Sigma^n(2) \leq (S^\lambda[n])^{\perp_n}$, and so $x \in (S^\lambda[n])^{\perp_n}$ for all sufficiently large n. Therefore, since $(S^\lambda)^\perp = \bigcap_{n \geq n_0} [(S^\lambda[n])^{\perp_n} \oplus M^\lambda[\Omega \setminus \{1, 2, \ldots, n\}]]$, we have that $x \in (S^\lambda)^\perp$ i.e. $(S^\lambda)^\perp \geq \Sigma(2)$.

Now let $x \in (S^\lambda)^\perp$. Then $x \in \Sigma^n(1) + \Sigma^n(2)$ for some sufficiently large n. We want to show that $x \in \Sigma^n(2)$, because then $x \in \Sigma(2)$ since $\Sigma^n(2)$ is contained in $\Sigma(2)$.

So assume, for a contradiction, that $x \in (\Sigma^n(1) + \Sigma^n(2)) \setminus \Sigma^n(2)$ and x is of minimal weight. Let x^- be the sum of all those tabloids involved in x whose entries in the bottom $r - 1$ rows are elements of $\{1, 2, \ldots, \lambda_2 + \cdots + \lambda_r\}$. We take the coefficient of such a tabloid in x^- to be the same as its coefficient in x. Without loss of generality, we can assume that $x^- \neq 0$ (because otherwise we can take a suitable translate of x), and so $\mathit{weight}(x - x^-) < \mathit{weight}(x)$. Let $A(x) := \bigcup_{\{u\} \in Supp(x)} \xi(\{u\})$, where ξ maps a tabloid to the subset of Ω consisting of the entries in the bottom $r - 1$ rows of the tabloid. Then $A(x)$ is a finite subset of Ω, so we choose λ_2 elements $a_1, \ldots, a_{\lambda_2}$ of $\Omega \setminus A(x)$. Now let s be any λ-tableau whose entries in the bottom $r - 1$ rows are elements of $\{1, 2, \ldots, \lambda_2 + \cdots + \lambda_r\}$ and the first λ_2 entries in the top row are $a_1, \ldots, a_{\lambda_2}$. Then the polytabloid $e_s \in S^\lambda$ and so $\langle x, e_s \rangle = 0$ (since $x \in (S^\lambda)^\perp$). Now let

$$f_s = \{s\} \sum \left\{ (\operatorname{sgn} \sigma)\sigma : \sigma \in C_s \text{ and } \sigma \text{ fixes } a_1, \ldots, a_{\lambda_2} \right\}.$$

Now if $\sigma \in C_s$ is such that σ moves at least one of $a_1, \ldots, a_{\lambda_2}$, then the tabloid $\{s\}\sigma$ will have at least one element of $\{a_1, \ldots, a_{\lambda_2}\}$ in its bottom $r - 1$ rows. Since $a_1, \ldots, a_{\lambda_2}$ were chosen to be in $\Omega \setminus A(x)$, we have that $\{s\}\sigma$ is not involved in x, and so does not contribute to $\langle x, e_s \rangle$. Thus $\langle x, f_s \rangle = 0$, and moreover $\langle x^-, f_s \rangle = 0$ (because $\langle x - x^-, f_s \rangle = 0$).

Now let $\mu = (\lambda_2, \ldots, \lambda_r)$, and then for any λ-tabloid $\{t_1\} \in M^\lambda$, let $\widehat{\{t_1\}}$ be the μ-tabloid formed by deleting the top row of $\{t_1\}$. Extend this linearly to any element of M^λ: if $v \in M^\lambda$, $v = \sum d_t\{t\}$ then $\hat{v} = \sum d_t\{\hat{t}\}$.

Then $\hat{f_s} \in S^{(\lambda_2, \ldots, \lambda_r)}$, and so, since $\hat{s}$ can vary over all μ-tableaux with entries from $\{1, 2, \ldots, \lambda_2 + \cdots + \lambda_r\}$, we have that $\widehat{x^-} \in (S^\mu)^{\perp_\mu}$, where $\perp_\mu$ denotes orthogonality in M^μ. So $x^- \in \Sigma^n(2)$ (by Theorem 2.6). Thus $x - x^- \in (\Sigma^n(1) + \Sigma^n(2)) \setminus \Sigma^n(2)$ and $weight(x - x^-) < weight(x)$ which contradicts the minimality of $weight(x)$. Thus $x \in \Sigma(2)$, which completes the proof. ∎

The case when λ has two rows will be dealt with in a more straightforward way in the next section.

3. THE MODULES $F[\Omega]^k$

3.1. *k-Sets and β-Maps*

We shall now turn our attention to a particular class of partitions. Again we will take Ω to represent any infinite set (and again, without loss of generality, we will treat Ω as if it were the set of natural numbers). We let λ be the partition of Ω with two parts, one of which is finite of size k, that is, $\lambda = (\infty - k, k)$, and let F be any field. We denote by $[\Omega]^k$ the set of all *k-sets* of Ω, that is, the set of all subsets of Ω of cardinality k. Then $Sym(\Omega)$ acts in the natural way on $[\Omega]^k$:

$$\{a_1, a_2, \ldots, a_k\}g = \{a_1 g, a_2 g, \ldots, a_k g\} \qquad \forall\, g \in Sym(\Omega).$$

This action turns $F[\Omega]^k$, the vector space over F with basis consisting of the elements of $[\Omega]^k$, into an $FSym(\Omega)$-module.

Now consider $M^{(\infty - k, k)}$ defined over F. It is easy to see that any tabloid in $M^{(\infty - k, k)}$ is uniquely determined by its bottom row, and so there is an obvious $FSym(\Omega)$-isomorphism between $M^{(\infty - k, k)}$ and $F[\Omega]^k$, given by mapping a tabloid $\{t\}$ to a k-set consisting of the k distinct elements of Ω in the bottom row of $\{t\}$. Thus, from now on, we shall not distinguish between k-sets from $[\Omega]^k$ and tabloids of $M^{(\infty - k, k)}$. When explicitly writing tabloids, we shall frequently omit the top row. We shall denote the Specht module of $F[\Omega]^k$ by S^k (so $S^k \cong S^{(\infty - k, k)}$).

In the previous section, we gave a description of the module orthogonal to the Specht module for infinite partitions with more than two parts, and we now investigate what happens for infinite partitions with precisely two parts. The following result not only tells us about $(S^k)^\perp$, but also gives us information about the reducibility of S^k.

LEMMA 3.1. $(S^k)^\perp = \{0\}$.

Proof. Assume, for a contradiction, that $(S^k)^\perp \neq \{0\}$. Let $x \in (S^k)^\perp$, $x \neq 0$. So the inner product of x with any element of S^k is zero. We will construct an element of S^k which has nonzero inner product with x, giving the required contradiction.

Since $x \neq 0$, there is a tabloid in $Supp(x)$, say $\overline{i_1 \cdots i_k}$ (recall that a tabloid is identified by its bottom row). We can assume without loss of generality that the coefficient of this tabloid in x is 1.

Now define $\Sigma(x) := \bigcup_{\{t\} \in Supp(x)} \xi(\{t\})$ where $\xi(\overline{a_1 \cdots a_k}) = \{a_1, \ldots, a_k\}$ (i.e. ξ can be thought of as the isomorphism between $M^{(\infty - k, k)}$ and $F[\Omega]^k$). Now x is a finite linear combination of tabloids, thus $\Sigma(x)$ is a finite subset of the infinite set Ω. So we can choose $j_1, \ldots, j_k \in \Omega \setminus \Sigma(x)$. Now let t be the tableau whose bottom k entries are $i_1, \ldots, i_k$, and the first k entries in the top row are $j_1, \ldots, j_k$, i.e.

$$t = \begin{matrix} j_1 & \cdots & j_k & * * * \cdots \\ i_1 & \cdots & i_k & \end{matrix}$$

Then let $y = e_t$, so $y \in S^k$. By construction, the only tabloid the supports of x and y have in common is $\overline{i_1 \cdots i_k}$, and this appears in both x and y with coefficient 1, therefore $\langle x, y \rangle = 1$, which is a contradiction. ∎

Combining this with the submodule theorem, we have:

THEOREM 3.2. *Any nonzero submodule of $F[\Omega]^k$ contains the Specht module, S^k.*

As an obvious corollary we have:

COROLLARY 3.3. *The Specht module S^k of $F[\Omega]^k$ is irreducible.*

Again, there exist the natural $FSym(\Omega)$-homomorphisms between the $F[\Omega]^k$, as described earlier for M^λ, but we adopt a change of notation:

DEFINITION 3.4. If $0 \leq j < k$, there is a natural $FSym(\Omega)$-homomorphism $\beta_{k,j}$: $F[\Omega]^k \to F[\Omega]^j$ given by $\beta_{k,j}(w) = \Sigma\{w' : w' \in [w]^j\}$ for $w \in [\Omega]^k$ and extended linearly.

Note that this homomorphism coincides with the homomorphism $\hat{\psi}_{1, k-j}$ when considering the partition $(\infty - k, k)$.

COROLLARY 3.5. *Let* $0 \leq l < k$. *Then* $F[\Omega]^k/\ker \beta_{k,l}$ *has a unique minimal submodule isomorphic to* S^l.

Proof. We have that $F[\Omega]^k/\ker \beta_{k,l} \cong \operatorname{im} \beta_{k,l}$, and $\operatorname{im} \beta_{k,l} \leq F[\Omega]^l$. Now $\beta_{k,l}$ is nonzero and so since S^l is the unique minimal submodule of $F[\Omega]^l$, then $\operatorname{im} \beta_{k,l}$ has a unique minimal submodule S^l. ∎

Using this new notation, we have immediately from Theorem 2.4:

THEOREM 3.6 (The Intersection Theorem for k-Sets). *For* $0 \leq i < k$, *let* $\beta_{k,i}$ *be the* β*-maps as defined above. Then we can characterize the Specht module of* $F[\Omega]^k$ *as*

$$S^k = \bigcap_{i=0}^{k-1} \ker \beta_{k,i}.$$

Removing kernels one at a time from this intersection produces a chain of submodules of $F[\Omega]^k$. It turns out that this is precisely the correct thing to do to find all submodules of $F[\Omega]^k$.

3.2. *Composition Series for* $F[\Omega]^k$

Firstly a general fact about finite composition series (see, for example, 13.7 in [2]):

THEOREM 3.7 (Jordan–Hölder). *Let R be any ring. If an R-module M possesses a finite composition series, then any two composition series of M are equivalent* (*where equivalent means that any two composition series have the same number of factors, and the factors can be paired off in such a way that the corresponding factors are R-isomorphic*).

We shall require that for partitions $(\infty - k, k)$ and $(\infty - l, l)$ of Ω, where $k \neq l$, the corresponding Specht modules S^k and S^l are not $FSym(\Omega)$-isomorphic. From 11.3 in [6] we have:

LEMMA 3.8. *Let n be any finite natural number. If* $\lambda = (n-k, k)$, $\mu = (n-l, l)$ *are partitions of n* (*so* $2k < n$, $2l < n$) *then* $S^\lambda \cong S^\mu \Rightarrow \lambda = \mu$.

We can use this to deduce the same result for infinite partitions with two parts. Before we state and prove this result, we need some notation. Recall that $[\Omega]^k$ denotes the set of all k-sets of Ω. If $n \in \mathbf{N}$, denote the set of all k-sets of $\{1, 2, \ldots, n\}$ by $[n]^k$. We note that $F[n]^k \cong M^{(n-k,k)}$. Let $S^k[n]$ denote the Specht module of $F[n]^k$, so $S^k[n] = S^k \cap F[n]^k$, where S^k is of course the Specht module of $F[\Omega]^k$. More generally, if $U \leq F[\Omega]^k$, let $U[n]$ be that $FSym(n)$-submodule of $F[n]^k$ given by $U[n] = U \cap F[n]^k$.

Let $G[n] = Sym(\{1, 2, \dots, n\}) \times Sym(\Omega \setminus \{1, 2, \dots, n\})$, so $G[n]$ is the setwise stabilizer of $\{1, 2, \dots, n\}$ in $Sym(\Omega)$. Note that $U[n]$, defined above, is also an $FG[n]$-submodule of $F[\Omega]^k$.

PROPOSITION 3.9. *Let M_i be an $FSym(\Omega)$-submodule of $F[\Omega]^k$. Then for $n \in \mathbf{N}$, M_i has a largest finite-dimensional $G[n]$-submodule, namely $M_i[n]$.*

Proof. Suppose, for a contradiction, that U is a finite-dimensional $G[n]$-submodule of M_i with $M_i[n] < U$. So there exists a tabloid $\{\tau\}$ involved in an element of $U \setminus M_i[n]$ with one of the bottom row entries of $\{\tau\}$ greater than n. Without loss of generality, to simplify notation, we shall assume $\{\tau\} = \overline{1\,2 \cdots\, k-1\; n+1}$. Now since U is $G[n]$-invariant, $\{\tau\}\alpha \in U$ for all $\alpha \in G[n]$. Now any element of $G[n]$ has the form gh, where $g \in Sym(\{1, 2, \dots, n\})$ and $h \in Sym(\Omega \setminus \{1, 2, \dots, n\})$. The permutation g will permute the first $k-1$ entries in the bottom row of $\{\tau\}$, and h takes the final entry, which is $n+1$, to another element of $\Omega \setminus \{1, 2, \dots, n\}$ (providing h is nontrivial). Thus, if h is nontrivial $\{\tau\}gh$ gives another distinct tabloid of U, and since $Sym(\Omega \setminus \{1, 2, \dots, n\})$ is infinite, the set $\{\{\tau\}\alpha\colon \alpha \in G[n]\}$ is a set of infinitely many distinct tabloids of U. These tabloids must be involved in the basis elements of U. But now any basis element of U has finite support and so there must be infinitely many basis elements of U. This contradicts U being finite-dimensional. ∎

PROPOSITION 3.10. *Let $\lambda = (\infty - k, k)$ and $\mu = (\infty - l, l)$ be two distinct partitions of Ω, i.e. $k \neq l$. Then the Specht modules S^k and S^l of $M^{(\infty - k, k)}$ and $M^{(\infty - l, l)}$ respectively are not $FSym(\Omega)$-isomorphic.*

Proof. Let $n > \max\{2k, 2l\}$ and consider $G[n]$ acting on S^λ. Then S^λ has a maximal finite-dimensional $G[n]$-submodule $S^\lambda[n]$ (by Proposition 3.9). Similarly, S^μ has a maximal finite-dimensional $G[n]$-submodule $S^\mu[n]$. Now if $S^\lambda[n]$ and $S^\mu[n]$ are $FG[n]$-isomorphic, then $S^{(n-k,k)}$ and $S^{(n-l,l)}$ are $FSym(n)$-isomorphic (where, of course, $Sym(n)$ denotes $Sym(\{1, 2, \dots, n\})$). So now if $S^\lambda \cong S^\mu$ (as $FSym(\Omega)$-modules), then $S^\lambda[n]$ and $S^\mu[n]$ are $FG[n]$-isomorphic, and so $S^{(n-k,k)}$ and $S^{(n-l,l)}$ are $FSym(n)$-isomorphic, and therefore, by Lemma 3.8, $(n-k, k) = (n-l, l)$. That is, $k = l$ and so $\lambda = \mu$. ∎

From now on we shall relax the notation and drop "$FSym(\Omega)$-" where the context is clear i.e. $FSym(\Omega)$-isomorphic becomes isomorphic, etc. We now show that $F[\Omega]^k$ does indeed have a composition series; moreover it is shown that each composition factor is isomorphic to a Specht module, and so by the above result we can distinguish between nonisomorphic factors.

PROPOSITION 3.11. *$F[\Omega]^k$ has a finite composition series in which each factor is isomorphic to a Specht module S^l, for some $l \leq k$, with each S^l appearing at least once, and S^k appearing exactly once.*

Proof. Use induction on k. Firstly note that $S^0 = F[\Omega]^0 \cong F$, the trivial module, which completes the proof of the base step of the induction.

Now for each $l < k$, assume $F[\Omega]^l$ has a finite composition series in which each factor is isomorphic to a Specht module S^m, for some $m \leq l$, with each S^m appearing at least once, and S^l appearing exactly once (so by the Jordan–Hölder theorem, *every* composition series of $F[\Omega]^l$ has this property).

Then by Theorem 3.6, $F[\Omega]^k / S^k$ is isomorphic to a submodule of $\bigoplus_{0 \leq l < k} F[\Omega]^l$ (consider the map which takes $x \in F[\Omega]^k$ to $(\beta_{k,0}(x), \dots, \beta_{k,k-1}(x))$). By the induction hypothesis, $\bigoplus_{0 \leq l < k} F[\Omega]^l$ has a finite composition series with each factor isomorphic to S^l for some $l < k$, so by Theorem 3.7, every composition series of $\bigoplus_{0 \leq l < k} F[\Omega]^l$ has this property. Therefore any submodule of $\bigoplus_{0 \leq l < k} F[\Omega]^l$ also has this property. As S^k is irreducible, it follows that $F[\Omega]^k$ has a finite composition series where S^k appears once, and the other composition factors are isomorphic to S^l, for $l < k$. Each S^l appears at least once by Corollary 3.5. ∎

DEFINITION 3.12. Now let λ be a partition of any set Λ. Then a series of submodules of M^λ is called a *Specht series* if each factor is isomorphic to a Specht module.

The only result we shall require about Specht series is the following, again due to James (see 17.17 in [6]). It concerns the existence of a particular Specht series:

LEMMA 3.13. *When $m \leq n - m$, $M^{(n-m,m)}$ has a Specht series with factors $S^{(n)}, S^{(n-1,1)}, S^{(n-2,2)}, \dots, S^{(n-m,m)}$ reading from the top.*

In the next result, we make use of the notation introduced before Proposition 3.10, which enables us to restrict our infinite-dimensional modules to finite-dimensional ones.

LEMMA 3.14. *Let $0 = M_0 < M_1 < \cdots < M_{r-1} < M_r = F[\Omega]^k$ be a series of submodules of $F[\Omega]^k$ such that $M_i/M_{i-1} \cong S^{l_i}$ for $i = 1, \dots, r$. Then if n is sufficiently large, $F[n]^k$ has a Specht series $0 = M_0[n] < M_1[n] < \cdots < M_{r-1}[n] < M_r[n] = F[n]^k$ with $M_i[n]/M_{i-1}[n] \cong S^{l_i}[n]$ for $i = 1, \dots, r$.*

Proof. Firstly, for $i = 1, 2, \dots, r$, let $\gamma_i \colon M_i/M_{i-1} \to S^{l_i}$ denote the isomorphisms in the hypothesis. Then for $i = 1, 2, \dots, r$ let $v_i \in M_i \setminus M_{i-1}$ be such that $\gamma_i(v_i + M_{i-1})$ is a polytabloid, and choose n large enough so that for all $i = 1, 2, \ldots, r$ we have $v_i \in F[n]^k$. Then $v_i \in M_i[n] \setminus M_{i-1}[n]$ (for $i = 1, 2, \ldots, r$), so we have the following chain of submodules of $F[n]^k$:

$$0 = M_0[n] < M_1[n] < \cdots < M_{r-1}[n] < M_r[n] = F[n]^k. \qquad (1)$$

Now for $i = 1, 2, \ldots, r$ define $\gamma_i^*\colon M_i[n]/M_{i-1}[n] \to S^{l_i}$ by $\gamma_i^*(x_i + M_{i-1}[n]) = \gamma_i(x_i + M_{i-1})$ for all $x_i \in M_i[n]$. Clearly γ_i^* is well defined (because γ_i is).

CLAIM. (i) γ_i^* *is an* $FG[n]$*-homomorphism*; (ii) γ_i^* *is injective*; (iii) $\operatorname{im} \gamma_i^* = S^{l_i}[n]$.

Proof of Claim. (i) For all $x_i \in M_i[n]$ and for all $g \in G[n]$ we have

$$\begin{aligned}
&\gamma_i^*(x_i + M_{i-1}[n])g \\
&\quad = \gamma_i(x_i + M_{i-1})g \\
&\quad = \gamma_i((x_i + M_{i-1})g) \\
&\quad = \gamma_i(x_i g + M_{i-1}) && \left(\text{since } M_{i-1} \le F[\Omega]^k\right) \\
&\quad = \gamma_i^*(x_i g + M_{i-1}[n]) && (\text{on noting that } x_i g \in M_i[n]) \\
&\quad = \gamma_i^*((x_i + M_{i-1}[n])g) && (M_{i-1} \text{ is an } FG[n]\text{-module}),
\end{aligned}$$

which proves that γ_i^* is an $FG[n]$-homomorphism.

(ii) Let $x_i, y_i \in M_i[n]$ be such that $\gamma_i^*(x_i + M_{i-1}[n]) = \gamma_i^*(y_i + M_{i-1}[n])$. Then $\gamma_i(x_i + M_{i-1}) = \gamma_i(y_i + M_{i-1})$, so we have that $x_i + M_{i-1} = y_i + M_{i-1}$ since γ_i is injective. Thus $x_i - y_i \in M_{i-1}$. But also we have $x_i - y_i \in M_i[n]$, and so $x_i - y_i \in M_i[n] \cap M_{i-1} = M_{i-1}[n]$. Thus $x_i + M_{i-1}[n] = y_i + M_{i-1}[n]$ and so γ_i^* is injective.

(iii) Recall that v_i was chosen such that $\gamma_i(v_i + M_{i-1})$ is a polytabloid. Now $v_i + M_{i-1}[n]$ lies in a finite $G[n]$-orbit and so $\gamma_i^*(v_i + M_{i-1}[n])$ also lies in a finite $G[n]$-orbit. Thus we have that $\gamma_i^*(v_i + M_{i-1}[n]) \in F[n]^{l_i} \cap S^{l_i} = S^{l_i}[n]$. So $\operatorname{im} \gamma_i^* \le S^{l_i}[n]$. Now $\gamma_i(v_i + M_{i-1})$ is a polytabloid, so $\gamma_i^*(v_i + M_{i-1}[n])$ is a polytabloid, and so $\langle \gamma_i^*(v_i + M_{i-1}[n]) \rangle_{FG[n]} = S^{l_i}[n]$. Thus $\operatorname{im} \gamma_i^* = S^{l_i}[n]$. This completes the proof of the claim.

So we have that γ_i^* is a well-defined bijective $FG[n]$-homomorphism from $M_i[n]/M_{i-1}[n]$ to $S^{l_i}[n]$. This completes the proof of the lemma. ∎

We are now in a position to prove the main result, which gives us the composition factors of $F[\Omega]^k$.

THEOREM 3.15. *$F[\Omega]^k$ has a composition series in which the composition factors are isomorphic to S^l, where $l = 0, 1, \ldots, k$, and each S^l appears exactly once.*

Proof. We know that any composition series of $F[\Omega]^k$ has each factor isomorphic to S^l, for some $l \le k$, with each l appearing at least once. Let $0 < M_1 < M_2 < \cdots < M_{r-1} < M_r = F[\Omega]^k$ be such a composition series, so $M_i/M_{i-1} \cong S^{l_i}$ for some $l_i \le k$ (and each l_i appearing at least once). Then by the previous lemma, for large enough n, the chain of $FG[n]$-submodules $0 < M_1[n] < \cdots < M_{r-1}[n] < M_r[n] = F[n]^k$ is such that $M_i[n]/M_{i-1}[n] \cong S^{l_i}[n]$ for all i.

Now each $M_i[n]$ is a finite-dimensional $FG[n]$-submodule of $F[n]^k$, so

$$\dim\left(F[n]^k\right) = \dim\left(\frac{F[n]^k}{M_{r-1}[n]}\right) + \dim\left(\frac{M_{r-1}[n]}{M_{r-2}[n]}\right) + \cdots + \dim\left(\frac{M_1[n]}{0}\right),$$

and so we have

$$\dim\left(F[n]^k\right) = \dim\left(S^{l_r}[n]\right) + \dim\left(S^{l_{r-1}}[n]\right) + \cdots + \dim\left(S^{l_1}[n]\right). \quad (2)$$

But now $F[n]^k$ has a Specht series (for $k < n - k$) $S^0[n], S^1[n], \ldots, S^k[n]$ (see Lemma 3.13), and so $\dim(F[n]^k) = \dim(S^0[n]) + \dim(S^1[n]) + \cdots + \dim(S^k[n])$. Thus in (2), $\dim(S^l[n])$ appears at most once, for every $l \le k$. But already in (2) each $S^l[n]$ appears at least once, and so each $S^l[n]$ appears exactly once. i.e. $\{l_0, l_1, \ldots, l_r\} = \{0, 1, \ldots, k\}$ and so $r = k$. Thus our original composition series for $F[\Omega]^k$ has k factors, namely $S^0, S^1, \ldots, S^k$. ∎

Remarks. (1) By the Jordan–Hölder theorem, *every* composition series of $F[\Omega]^k$ has length $k + 1$ and the factors of any composition series are (in no particular order) $S^0, S^1, \ldots, S^{k-1}, S^k$.

(2) Now we have the existence of a composition series of $F[\Omega]^k$, what Corollary 3.5 tells us is that for l satisfying $0 \le l < k$, every composition series of $F[\Omega]^k/\ker\beta_{k,l}$ has a bottom composition factor S^l.

COROLLARY 3.16. *Let $U < F[\Omega]^k$ and let l be an integer satisfying $0 \le l < k$. Then S^l is a composition factor of $F[\Omega]^k/U$ if and only if $U \le \ker\beta_{k,l}$.*

Proof. Assume $U \le \ker\beta_{k,l}$. Now by remark (2) above, we know that S^l is the unique bottom composition factor of $F[\Omega]^k/\ker\beta_{k,l}$. Clearly the composition factors of $F[\Omega]^k/\ker\beta_{k,l}$ appear amongst the composition factors of $F[\Omega]^k/U$, thus S^l is a composition factor of $F[\Omega]^k/U$.

Conversely, assume that $U \not\le \ker\beta_{k,l}$. Then $(U + \ker\beta_{k,l})/\ker\beta_{k,l}$ is nonzero, so it has a unique bottom composition factor S^l by remark (2) above. But now $(U + \ker \beta_{k,l})/\ker \beta_{k,l} \cong U/(U \cap \ker \beta_{k,l})$, and so by Theorem 3.15, S^l is not a composition factor of $F[\Omega]^k/U$. ∎

We can now give a precise description of the submodules of $F[\Omega]^k$:

COROLLARY 3.17. *Let $U < F[\Omega]^k$. Then $U = \bigcap\{\ker \beta_{k,l}\colon S^l$ is a composition factor of $F[\Omega]^k/U\}$.*

Proof. Assume the composition factors of $F[\Omega]^k/U$ are $S^{l_1}, \ldots, S^{l_r}$. Then by the previous corollary, $U \leq \ker \beta_{k,l_i}$ for $i = 1, 2, \ldots, r$ and so $U \leq \bigcap_{i=1}^r \ker \beta_{k,l_i}$. But now if $U < \bigcap_{i=1}^r \ker \beta_{k,l_i}$, then $\bigcap_{i=1}^r \ker \beta_{k,l_i}/U$ has S^{l_t} as a composition factor, with $t \in \{1, 2, \ldots, r\}$, which implies that S^{l_t} is not a composition factor of $F[\Omega]^k/\bigcap_{i=1}^r \ker \beta_{k,l_i}$, which means that S^{l_t} is not a composition factor of $F[\Omega]^k/\ker \beta_{k,l_t}$, which is a contradiction. Thus $U = \bigcap_{i=1}^r \ker \beta_{k,l_i}$. ∎

Remark. What this result tells us is that every submodule of $F[\Omega]^k$ is an intersection of some of the kernels $\ker \beta_{k,l}$ where $0 \leq l < k$.

COROLLARY 3.18. *The maximal proper submodules of $F[\Omega]^k$ are kernels of the β-maps.*

Proof. Let V be a maximal proper submodule of $F[\Omega]^k$, so $F[\Omega]^k/V \cong S^l$, for some $l \leq k$. Thus, by Corollary 3.17, $V = \ker \beta_{k,l}$. ∎

With the aid of the next technical result, we can give a result about the surjectivity of the β-maps.

PROPOSITION 3.19. *Let F be a field of characteristic p. Let $0 \leq m < l < k$, and consider the maps $\beta_{k,l}\colon F[\Omega]^k \to F[\Omega]^l$ and $\beta_{l,m}\colon F[\Omega]^l \to F[\Omega]^m$. Then* $\operatorname{im} \beta_{k,l} \leq \ker \beta_{l,m}$ *if and only if p divides $\binom{k-m}{l-m}$.*

Proof. Let $x \in [\Omega]^k$, and let $y = \beta_{k,l}(x)$. Then $y \in \operatorname{im} \beta_{k,l}$. Now $\beta_{l,m}$ maps y to a sum of m-sets. Now the coefficient of a typical m-set z in $\beta_{l,m}(y)$ is equal to the number of ways of choosing an l-set containing z from x, which is $\binom{k-m}{l-m}$. Thus if p divides $\binom{k-m}{l-m}$ then $y \in \ker \beta_{l,m}$ so $\operatorname{im} \beta_{k,l} \leq \ker \beta_{k,m}$, and if p does not divide $\binom{k-m}{l-m}$ then $y \notin \ker \beta_{l,m}$ so $\operatorname{im} \beta_{k,l} \nleq \ker \beta_{l,m}$. ∎

THEOREM 3.20. *Let F be a field of characteristic p. Let $0 \leq l < k$, and consider the map $\beta_{k,l}\colon F[\Omega]^k \to F[\Omega]^l$. Then $\beta_{k,l}$ is onto if and only if p does not divide $\binom{k}{l}\binom{k-1}{l-1}\cdots\binom{k-l+1}{1}$.*

Proof. We know that $\beta_{k,l}$ is onto if and only if im $\beta_{k,l} = F[\Omega]^l$. Now since the maximal proper submodules of $F[\Omega]^l$ are kernels of β-maps,

$$\begin{aligned}\operatorname{im} \beta_{k,l} = F[\Omega]^l &\Leftrightarrow \operatorname{im} \beta_{k,l} \nleq \ker \beta_{l,m}, \qquad 0 \le m < l,\\ &\Leftrightarrow p \text{ does not divide } \binom{k-m}{l-m}, \qquad m = 0, 1, \dots, l-1,\\ &\Leftrightarrow p \text{ does not divide } \binom{k}{l}\binom{k-1}{l-1}\cdots\binom{k-l+1}{1}. \qquad \blacksquare\end{aligned}$$

Remarks. (1) This product of binomial coefficients can be simplified to a product of fractions: $\beta_{k,l}$ is onto if and only if p does not divide $\prod_{i=0}^{l-1}((k-i)/(l-i))^{i+1}$.

(2) From the above two proofs we see that $\beta_{k,l}$ is onto if and only if all the binomial coefficients $\binom{k-m}{l-m}$ for $m = 0, 1, \dots, l-1$ are nonzero in the field. Thus when the field F has characteristic zero, the map $\beta_{k,l}\colon F[\Omega]^k \to F[\Omega]^l$ is always onto.

From this theorem we can obtain a corresponding result involving the finite β-maps:

COROLLARY 3.21. *If p does not divide $\binom{k}{l}\binom{k-1}{l-1}\cdots\binom{k-l+1}{1}$ then, for large enough n, the map $\beta^n_{k,l}\colon F[n]^k \to F[n]^l$ is onto. Conversely, if $\beta^n_{k,l}$ is onto for sufficiently large enough n, then p does not divide $\binom{k}{l}\binom{k-1}{l-1}\cdots\binom{k-l+1}{1}$.*

Proof. Suppose p does not divide $\binom{k}{l}\binom{k-1}{l-1}\cdots\binom{k-l+1}{1}$. Then the β-map $\beta_{k,l}\colon F[\Omega]^k \to F[\Omega]^l$ is onto. Let $w \in [\Omega]^l$. Then there exists an $x \in F[\Omega]^k$ such that $\beta_{k,l}(x) = w$. Now choose $n_0 \in \mathbf{N}$ large enough so that $Supp(x) \subseteq [n]^k$ for all $n \ge n_0$. Then, by definition, $\beta^n_{k,l}(x) = w$, so $w \in [n]^l$. So $\beta^n_{k,l}\colon F[n]^k \to F[n]^l$ is onto for all $n \ge n_0$.

Conversely, suppose p divides $\binom{k}{l}\binom{k-1}{l-1}\cdots\binom{k-l+1}{1}$. Then $\beta_{k,l}$ is not onto. So there exists $w \in F[\Omega]^l$ such that there is no element $x \in F[\Omega]^k$ with $\beta_{k,l}(x) = w$. Let n_1 be large enough so that $Supp(w) \subseteq [n]^l$ and $Supp(x) \subseteq [n]^k$ for all $n \ge n_1$. Then $\beta^n_{k,l}\colon F[n]^k \to F[n]^l$ is not onto for all $n \ge n_1$. $\blacksquare$

Remark. It would be interesting to have an explicit bound for n_0 here.

We conclude this section with a result regarding the order in which the factors appear in a particular composition series of $F[\Omega]^k$. To prove this, we need the following result:

PROPOSITION 3.22. *Let* K_j *denote* $\ker \beta_{k,j}$. *Suppose that*

$$K_{i(1)} \cap \cdots \cap K_{i(r)} = K_{i(1)} \cap \cdots K_{i(r)} \cap K_n$$

for some $i(1), \dots, i(r)$, $n \in \{0, 1, \dots, k-1\}$. *Then* $K_{i(j)} \leq K_n$ *for some* j.

Proof. Suppose, for a contradiction, that $K_{i(j)} \nleq K_n$ for any j. So for all j, we have that S^n is not a composition factor of $F[\Omega]^k / K_{i(j)}$ (by Corollary 3.16). As $F[\Omega]^k / (K_{i(1)} \cap \cdots \cap K_{i(r)})$ embeds into $\bigoplus_{j \leq r} F[\Omega]^k / K_{i(j)}$, it follows that S^n is not a composition factor of $F[\Omega]^k / (K_{i(1)} \cap \cdots \cap K_{i(r)})$, that is, S^n is not a composition factor of $F[\Omega]^k / (K_{i(1)} \cap \cdots \cap K_{i(r)} \cap K_n)$. But S^n is a composition factor of $F[\Omega]^k / K_n$ and $K_{i(1)} \cap \cdots K_{i(r)} \cap K_n \leq K_n$, so we have a contradiction. ∎

LEMMA 3.23. $F[\Omega]^k$ *has a composition series with factors in the order* (*from the top*) $S^0, S^1, \dots, S^k$, *and the corresponding composition series is*

$$0 < \bigcap_{l=0}^{k-1} \ker \beta_{k,l} < \bigcap_{l=0}^{k-2} \ker \beta_{k,l} < \cdots < \ker \beta_{k,0} \cap \ker \beta_{k,1}$$
$$< \ker \beta_{k,0} < F[\Omega]^k.$$

Proof. This result follows easily from Corollary 3.17 and Theorem 3.15 if we can show that, for any d satisfying $0 < d \leq k-1$, we have that the quotient module $\bigcap_{l=0}^{d-1} \ker \beta_{k,l} / \bigcap_{l=0}^{d} \ker \beta_{k,l}$ is nonzero.

So assume, for a contradiction, that $\bigcap_{l=0}^{d-1} \ker \beta_{k,l} = \bigcap_{l=0}^{d} \ker \beta_{k,l}$. Then by Proposition 3.22, $\ker \beta_{k,j} \leq \ker \beta_{k,d}$ for some j satisfying $0 \leq j < d$. So by Corollary 3.16, S^d is a composition factor of $F[\Omega]^k / \ker \beta_{k,j}$. But $F[\Omega]^k / \ker \beta_{k,j}$ is isomorphic to $\operatorname{im} \beta_{k,j} \leq F[\Omega]^j$, and $j < d$, so by Theorem 3.15, $F[\Omega]^k / \ker \beta_{k,j}$ cannot have a composition factor isomorphic to S^d. This is a contradiction. Thus we have that $\bigcap_{l=0}^{d-1} \ker \beta_{k,l} / \bigcap_{l=0}^{d} \ker \beta_{k,l}$ is nonzero.

In fact, by Corollary 3.17, we have that $\bigcap_{l=0}^{d-1} \ker \beta_{k,l} / \bigcap_{l=0}^{d} \ker \beta_{k,l}$ is isomorphic to S^d, and so by Theorem 3.15, we have the required composition series. ∎

4. SPECIAL CASES

Before we give the algorithm to compute the submodule structure of $F[\Omega]^k$, we look at some special cases where the submodule structure can be explicitly calculated independently of the algorithm. The results in this section make use of the following:

PROPOSITION 4.1. *Let F be a field of characteristic p (p a prime), and consider the maps $\beta_{k,i}\colon F[\Omega]^k \to F[\Omega]^i$ and $\beta_{k,j}\colon F[\Omega]^k \to F[\Omega]^j$. Then* $\ker \beta_{k,i} \leq \ker \beta_{k,j}$ *if and only if p does not divide $\binom{k-j}{i-j}$, where* $0 \leq j < i < k$.

Proof. Consider the composition of maps $\beta_{i,j}\beta_{k,i}$. As in Proposition 3.19, $\beta_{i,j}\beta_{k,i} = \binom{k-j}{i-j}\beta_{k,j}$. Thus if p does not divide $\binom{k-j}{i-j}$ then $\ker \beta_{k,i} \leq \ker \beta_{k,j}$.

Now assume that p does divide $\binom{k-j}{i-j}$ and suppose, for a contradiction, that $\ker \beta_{k,i} \leq \ker \beta_{k,j}$. Now since p divides $\binom{k-j}{i-j}$ we have that $\operatorname{im} \beta_{k,i} \leq \ker \beta_{i,j}$. Now S^j is a composition factor of $F[\Omega]^i/\ker \beta_{i,j}$, and so by Theorem 3.15, S^j is not a composition factor of $\operatorname{im} \beta_{k,i}$. Now clearly $\operatorname{im} \beta_{k,i} \neq 0$, and $\operatorname{im} \beta_{k,i} \cong F[\Omega]^k/\ker \beta_{k,i}$. Once again, we note that S^j is a composition factor of $F[\Omega]^k/\ker \beta_{k,j}$, and, by assumption, $\ker \beta_{k,i} \leq \ker \beta_{k,j}$. So S^j is a composition factor of $F[\Omega]^k/\ker \beta_{k,i}$, that is, S^j is a composition factor of $\operatorname{im} \beta_{k,i}$, which is a contradiction. ∎

4.1. *The Case* char $F = 0$

In [1], it is shown that when $\operatorname{char} F = 0$, the submodule structure of $F[\Omega]^k$ is uniserial with composition factors (from the top) $S^0, S^1, \ldots, S^k$, the corresponding composition series being $0 < \ker \beta_{k,k-1} < \ker \beta_{k,k-2} < \cdots < \ker \beta_{k,0} < F[\Omega]^k$. We now deduce this result from our results so far:

First we note that, as in the proof of Proposition 4.1, for $j = 0, 1, \ldots, k-2$ we have $\beta_{j+1,j}\beta_{k,j+1} = (k-j)\beta_{k,j}$, and so we have $0 < \ker \beta_{k,k-1} \leq \ker \beta_{k,k-2} \leq \cdots \leq \ker \beta_{k,0} \leq F[\Omega]^k$. Now let $U \leq F[\Omega]^k$, so $U = \cap\{\ker \beta_{k,j}\colon j \in J\}$ for some subset J of $\{0, 1, \ldots, k-1\}$. Then, by the nesting of the kernels, we have that $U = \ker \beta_{k,l}$ where l is the minimum element of J. Thus, by Corollary 3.17, the only nontrivial submodules of $F[\Omega]^k$ are $\ker \beta_{k,0}, \ker \beta_{k,1}, \ldots, \ker \beta_{k,k-1}$. But now $F[\Omega]^k$ has a composition series $0 < \ker \beta_{k,k-1} < \ker \beta_{k,k-2} < \cdots < \ker \beta_{k,0} < F[\Omega]^k$ (by Lemma 3.23) and this is clearly unique.

4.2. *The Case* char $F = p$, *where* $p > k$

If $p > k$, then p does not divide $\binom{k-j}{i-j}$ for all i and j satisfying $0 \leq i < j < k$, and so by Proposition 4.1, we have that $\ker \beta_{k,k-1} \leq \ker \beta_{k,k-2} \leq \cdots \leq \ker \beta_{k,0}$. But now, by Lemma 3.23, $F[\Omega]^k$ has a composition series $0 < \ker \beta_{k,k-1} < \ker \beta_{k,k-2} < \cdots < \ker \beta_{k,0} < F[\Omega]^k$, and since every submodule of $F[\Omega]^k$ is an intersection of kernels and all the kernels are nested, this is in fact the unique composition series of $F[\Omega]^k$. So when $\operatorname{char} F = p$, with $p > k$, $F[\Omega]^k$ is uniserial in exactly the same way as $F[\Omega]^k$ was for $\operatorname{char} F = 0$.

4.3. *The Case $k = p$, where $p = \operatorname{char} F$*

Here we let $k = p$, where $p = \operatorname{char} F$. Now p does not divide $\binom{k-j}{i-j}$ for all i and j satisfying $1 \le j < i < k$, so from Proposition 4.1 we have $\ker \beta_{k,k-1} \le \ker \beta_{k,k-2} \le \cdots \le \ker \beta_{k,1}$, and so $\bigcap_{l=1}^{k-1} \ker \beta_{k,l} = \ker \beta_{k,k-1}$. Using Proposition 4.1 with $p = k$, $i = k - 1$, and $j = 0$, we have $\ker \beta_{k,k-1} \nleq \ker \beta_{k,0}$.

Now $F[\Omega]^k/\ker \beta_{k,0} \cong \operatorname{im} \beta_{k,0} = S^0$, and $S^0 \cong F$, and so $\ker \beta_{k,0}$ has codimension 1 in $F[\Omega]^k$. Therefore the sum of submodules $\ker \beta_{k,0} + \ker \beta_{k,k-1}$ is either equal to $\ker \beta_{k,0}$ or $F[\Omega]^k$. But if $\ker \beta_{k,0} + \ker \beta_{k,k-1} = \ker \beta_{k,0}$ then $\ker \beta_{k,0}$ would have to contain $\ker \beta_{k,k-1}$, which is a contradiction. Thus we have $\ker \beta_{k,0} + \ker \beta_{k,k-1} = F[\Omega]^k$. Now we can make use of the results of Section 4.2: by the first isomorphism theorem $F[\Omega]^k/\ker \beta_{k,k-1} \cong \operatorname{im} \beta_{k,k-1}$, and by Proposition 3.19, $\operatorname{im} \beta_{k,k-1} = \ker \beta_{k-1,0}$. When $\operatorname{char} F = k$, $F[\Omega]^{k-1}$ is uniserial, as proved in the previous section, and so the composition factors of $F[\Omega]^k/\ker \beta_{k,k-1}$ are (from the top) $S^1, S^2, \dots, S^{k-1}$, and so the composition factors of $\ker \beta_{k-1,0}/S^k$ are $S^1, S^2, \dots, S^{k-1}$ (using the fact that $\ker \beta_{k,0} \cap \ker \beta_{k,k-1} = S^k$).

5. THE ALGORITHM

In this section we will present an algorithm to compute the submodule structure of $F[\Omega]^k$, for any nonnegative integer k. Throughout this section, it will be assumed that F is a field of characteristic p, since the submodule structure of $F[\Omega]^k$ is explicitly known when $\operatorname{char} F = 0$ (see Section 4). In Section 3.2 we proved that every submodule of $F[\Omega]^k$ was an intersection of kernels of the β-maps. In fact, since every kernel is a submodule of $F[\Omega]^k$ then every possible intersection of kernels is a submodule. So we know all the possible submodules—just compute all possible subsets of $\{0, 1, \dots, k-1\}$. Then for each such subset I, we have that $\bigcap_{i \in I} \ker \beta_{k,i}$ is a submodule of $F[\Omega]^k$. Thus we can identify each submodule of $F[\Omega]^k$ by a subset of $\{0, 1, \dots, k-1\}$. However, since we have that some kernels are contained in other kernels, some of the submodules are in fact the same. That is, we can have two different subsets of $\{0, 1, \dots, k-1\}$ which describe the same submodule. Thus we need a procedure which, given an intersection of kernels X (i.e., a subset of $\{0, 1, \dots, k-1\}$), determines all the kernels which could appear in the expression for X. So then, using this procedure, we can assign a unique subset of $\{0, 1, \dots, k-1\}$ to each submodule. Once we have all the submodules of $F[\Omega]^k$, each with its unique "label," it should then be clear how to construct the submodule lattice.

Before we present the algorithm, we will describe the "uniqueness" procedure. We are given an intersection of kernels as a subset J of $\{0, 1, \dots, k-1\}$. For each $j \in J$ we can calculate all the kernels which $\ker \beta_{k,j}$ is contained in, using Proposition 4.1. Thus we replace $\ker \beta_{k,j}$ by the intersection of all the kernels it is contained in. By Proposition 3.22, doing this will give us the desired description of the original kernel intersection.

We now present the "uniqueness" procedure (which we shall call "procedure unique") and the algorithm in a formal language, suitable for adaptation to computer programming languages.

Procedure Unique

input a subset I of $\{0, 1, \dots, k-1\}$
let tempI $= I$
for each $i \in I$ do
  for $j = 0$ to $i - 1$ do
    if p does not divide $\binom{k-j}{i-j}$ then add $\{j\}$ to tempI
  end
end
output the set tempI (which is also a subset of $\{0, 1, \dots, k-1\}$)

Algorithm to Compute All the Submodules of $F[\Omega]^k$

let $\mathcal{P}$ be the power set of $\{0, 1, \dots, k-1\}$
let temp $\mathcal{P} = \mathcal{P}$
for each J in $\mathcal{P}$ do
  apply Procedure unique to J (and let the output set be tempJ)
  let temp $\mathcal{P}$ = temp $\mathcal{P} - \{J\}$
  let temp $\mathcal{P}$ = temp $\mathcal{P} \cup$ {tempJ}
end

When the algorithm terminates, temp $\mathcal{P}$ will contain all the proper submodules of $F[\Omega]^k$ as subsets of $\{0, 1, \dots, k-1\}$, each one corresponding to an intersection of kernels.

ACKNOWLEDGMENTS

The material in this paper was written whilst the author was in receipt of an E.P.S.R.C. studentship. The author would also like to thank David Evans for his help and support, particularly in finding the composition series of length $k + 1$.

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